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A Survey of Direct Integration Methods in Structural Dynamics

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ABSTRACT

Several alternative methods for directly integrating the governing equations of motion of structural dynamics are reviewed. First, the characteristics of the matrix equations are examined (e.g.; the spread in structural eigenvalues, or stiffness; the bandwidth and sparseness; and the frequency spectrum of the forcing function). Then, the criteria that can be used to select a direct integration algorithm are discussed (e.g.; the artificial damping, the periodicity error). Emphasis is given to results obtained for the Houbolt, Newmark and Wilson operators, and their comparison to a class of stiffly stable operators. Recent application of these operators to nonlinear problems is discussed.

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I. Introduction

Due to the current importance of dynamic analysis for civil engineering structures, especially with regard to seismic loading, there has been a resurgence of interest in approximate methods for integrating the equations of motion. For many years the emphasis has been on the approximation of the spatial deformation of these structures, using more sophisticated finite element models, and including the effects of geometric and material nonlinearity [1]. Some of this emphasis is now being aimed toward the understanding of the characteristics of direct integration operators.

In order to be precise about the scope of this review, the distinction between forced structural vibration and wave propagation is made. Structural vibration problems are almost always dominated by low-frequency components of the response, since the energy requirements for exciting the higher modes are so severe. This is fortunate for the analyst, since the deterioration of the accuracy of the frequencies and mode shapes makes the computed results questionable for these higher modes. It would seem reasonable, therefore, not to spend needless time and effort trying to make the computed solution accurate with respect to these modes.

High-frequency response is very important for many wave propagation problems, however, especially where discontinuities in velocity or acceleration persist. If the response in the region around the discontinuity is of interest, the analyst will be forced to use a time step of integration so small that explicit integration methods, such as central-differencing with a lumped mass matrix, become attractive. It should be noted that wave propagation problems without velocity or acceleration discontinuities are handled quite nicely by the same techniques that are used for structural vibration [2].

In this review, several alternative methods for carrying out the step-by-step integration of the equations of motion of a structural system will be discussed. In the next section, the characteristics of general structural systems are emphasized so that the criteria for choosing a particular integration operator can be understood on these terms. Then, three of the most popular methods are analyzed and compared. Following that is a discussion of the relationship between these popular methods and a class of methods referred to in the literature as stiffly-stable [3]. Finally, recent results on the application of these results to nonlinear problems are summarized.

II. Structural System Characteristics

The governing equation of interest is given by

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{u}}(t) + \underset{\sim}{C}\,\dot{\underset{\sim}{u}}(t) + \underset{\sim}{K}\,\underset{\sim}{u}(t) = \underset{\sim}{F}(t) \quad , \qquad \text{(II.1)}$$

where $\underset{\sim}{M}$, $\underset{\sim}{C}$, and $\underset{\sim}{K}$ are the mass, damping, and stiffness matrices, respectively; $\ddot{\underset{\sim}{u}}(t)$, $\dot{\underset{\sim}{u}}(t)$, and $\underset{\sim}{u}(t)$ are the acceleration, velocity, and displacement vectors at time t , respectively; and $\underset{\sim}{F}(t)$ is the force vector. An incremental form of this equation, valid for elastic-plastic constitutive behavior and geometrically nonlinear behavior, has been developed in [4]. This incremental form is used, in conjunction with "residual load correction" in order to solve beam and axisymmetric shell dynamic problems. The governing equation (II.1) can be derived in a number of ways, such as by Rayleigh-Ritz-Galerkin or other finite element methods, but the mass matrix will always be assumed to be positive definite.

For most structural problems, the mass, damping, and stiffness matrices are sparse, banded, and symmetric. Attempts to solve (II.1), whether by direct integration (step-by-step) or by modal superposition, should take advantage of

these characteristics. The initial conditions for (II.1) involve the displacement and velocity at $t = 0$; therefore, the direct integration procedure requires at least this amount of information about the previous time, t_n , in order to predict the state of motion at the current time, t_{n+1} . Additional information about the state at time t_n , such as acceleration, or about the state of motion at an even earlier time (say t_{n-1}) leads to: (a) special starting procedures for the integration method; (b) extra storage requirements; and (c) extraneous solutions [5] that may create accuracy or instability problems. Unless ample justification for the extra information is given, the minimum amount should be used. To the writer's knowledge, only one such method has been proposed [6], the Gurtin Averaging operator, that is unconditionally stable; however, the accuracy of the method is low unless small time steps or negative damping is used.

The characteristics of (II.1) are best defined in terms of its natural frequencies and mode shapes. For a continuous system, the frequency spectrum ranges from the lowest, or fundamental, frequency up to an infinite limit point (there may be zero frequencies if rigid-body modes are present). For a discretized system, the infinite limit point no longer exists; instead, a frequency exists that corresponds to the most rapidly varying (in space) mode shape. This frequency is called the cut-off frequency. If the discretized system is excited by forcing functions having frequency content above the cut-off frequency, such as might be induced in a wave propagation problem, noise (random spatial response) is generated in the cut-off modal response.

Corresponding to each possible mode of vibration and frequency of the structural system is a period of vibration, or time constant, and the range of periodicity is a measure of the "stiffness" [3] of the system. The largest

period is associated with the fundamental mode and the smallest is associated with the cut-off frequency or, therefore, the "stiffest" system component.

As Jensen has pointed out [7], the integration of (II.1), which is, in general, a stiff system, presents the analyst with a dilemma. If the time step is reduced in order to accurately integrate the stiff components, then the step will be much too small for the longer period (lower frequency) responses, resulting in excessive computer time for the calculations. On the other hand, if the time step is chosen with regard for the low-frequency response, instability will result for explicit methods and integration error will give inaccurate solutions for the stiff components.

Four solutions of interest can be identified here: (a) the exact solution of the equations of motion of the continuous system; (b) the exact solution of (II.1), using infinite-precision arithmetic; (c) the exact solution of (II.1), using finite-precision arithmetic; and (d) the approximate solution of (II.1), using finite-precision arithmetic and a direct integration operator. Truncation error, such as the noise referred to above, represents error incurred from the truncation of a continuous to a discretized system and is the difference between (a) and (b). Round-off error depends on the degree of precision of the computational arithmetic and is the difference between (b) and (c). Integration error is the difference between (c) and (d).

The primary motive for studying various direct integration operators is to understand the nature of integration error, how to control it, and how to estimate the computing costs of this control. An integration operator [8] is defined as a transformation on the acceleration, velocity, and displacement vectors at time t_n (and, possibly, at earlier times) to the acceleration, velocity, and the displacement vectors at time t_{n+1}. If the mass, damping,

and stiffness matrices do not depend on the displacements, or their space and time derivatives, the transformation is said to be linear. If the transformation does not depend on the state at times earlier than t_n, it is called a single-step method; otherwise, it is a multi-step method.

Direct integration operators are generally written in the form

$$\underset{\sim}{K}_1 \, \bar{\underset{\sim}{u}}(t_{n+1}) = \underset{\sim}{F} + \underset{\sim}{K}_o \, \bar{\underset{\sim}{u}}(t_n, t_{n-1}, \ldots) \quad , \tag{II.2}$$

where $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_o$ are the matrices that define the transformation, $\underset{\sim}{F}$ is a vector of forcing functions, and

$$\bar{\underset{\sim}{u}}^T(t_{n+1}) = \left\langle \underset{\sim}{u}(t_{n+1}) \;\middle|\; \dot{\underset{\sim}{u}}(t_{n+1}) \;\middle|\; \ddot{\underset{\sim}{u}}(t_{n+1}) \right\rangle , \tag{II.3a}$$

$$\bar{\underset{\sim}{u}}^T(t_n, t_{n-1}, \ldots) = \left\langle \underset{\sim}{u}(t_n) \;\middle|\; \dot{\underset{\sim}{u}}(t_n) \;\middle|\; \ddot{\underset{\sim}{u}}(t_n) \;\middle|\; \underset{\sim}{u}(t_{n-1}) \;\middle|\; \cdots \right\rangle . \tag{II.3b}$$

The superscript T indicates the transpose of a vector or a matrix. The matrices $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_o$ are, in general, functions of the mass, damping and stiffness matrices, as well as the time step size, $\Delta t = t_{n+1} - t_n$. If the matrix $\underset{\sim}{K}_1$ can be put into upper or lower triangular form, the operator is said to be explicit; otherwise, it is implicit. The amplification matrix of the operator is defined by

$$\underset{\sim}{A} = \underset{\sim}{K}_1^{-1} \underset{\sim}{K}_o \quad , \tag{II.4}$$

assuming that an inverse exists. Then

$$\bar{\underset{\sim}{u}}(t_{n+1}) = \underset{\sim}{G} + \underset{\sim}{A}\bar{\underset{\sim}{u}}(t_n, t_{n-1}, \ldots) \quad , \tag{II.5}$$

where

$$\underset{\sim}{G} = \underset{\sim}{K}_1^{-1} \underset{\sim}{F} \ .$$

Stability of direct integration operators has been approached in two ways. Lax and Richtmyer [8] discuss the spectral radius of the amplification matrix; for structural dynamics, this spectral radius is defined in terms of the resonant structural frequencies of the system and the integration operator is stable for all time step sizes that cause the spectral radius to be bounded by unity. Dahlquist [9] has taken a different approach, introducing the concept of A-stability for systems of first-order ordinary differential equations. By this he means that the error introduced into the approximate solution by the particular integration method remains uniformly bounded for any time step size. This coincides with the Lax-Richtmyer notion of unconditional stability; i.e., the spectral radius of the operator is bounded by unity for all choices of time step size. Dahlquist has proven that linear multi-step methods are implicit if they are A-stable and that the "trapezoidal" rule has the smallest asymptotic error of any A-stable method.

III. Operator Comparisons

In this section the characteristics of the most popular direct integration operators are discussed and compared. The central difference operator, for example, has been shown to be conditionally stable [10] and, in addition, Krieg [11] has found that no explicit integration operator of order two has a stability region greater than the central difference operator. The Houbolt operator [12,13], on the other hand, is unconditionally stable [14] and has been compared to the central difference operator for accuracy and speed [15]. Both the Newmark [16] and Wilson Averaging [17,18] operators have also been shown to be unconditionally stable [6] and comparisons have been made with a precise integration operator [19] based on modal superposition. In spite of this work, however, some direct comparisons seem to be in order.

The Houbolt operator is obtained by fitting a cubic polynomial through values of the current displacement (to be found) and the three previous values. This necessitates a special starting procedure. Substituting into (II.1) the first and second derivatives of this polynomial, evaluated at the current time, results in an equation of motion

$$\begin{aligned}(\underset{\sim}{M} + \frac{11}{12}\Delta t\underset{\sim}{C} + \frac{1}{2}\Delta t^2\underset{\sim}{K})\underset{\sim}{u}_{n+1} &= \frac{1}{2}\Delta t^2\,\underset{\sim}{F}_{n+1} \\ &+ (\frac{5}{2}\underset{\sim}{M} + \frac{3}{2}\Delta t\underset{\sim}{C})\underset{\sim}{u}_n - (2\underset{\sim}{M} + \frac{3}{4}\Delta t\underset{\sim}{C})\underset{\sim}{u}_{n-1} \\ &+ (\frac{1}{2}\underset{\sim}{M} + \frac{1}{6}\Delta t\underset{\sim}{C})\underset{\sim}{u}_{n-2} \ .\end{aligned} \tag{III.1}$$

Following the stability procedure of von Neumann [20], let

$$\underset{\sim}{u}_n = \lambda^n \underset{\sim}{d} \ , \tag{III.2}$$

where $\underset{\sim}{d}$ is an arbitrary error vector. Then, the characteristic equation for λ is

$$(1 + \frac{1}{2}\xi + \frac{11}{12}\eta)\lambda^3 - (\frac{5}{2} + \frac{3}{2}\eta)\lambda^2 + (2 + \frac{3}{4}\eta)\lambda - (\frac{1}{2} + \frac{1}{6}\eta) = 0 \ , \tag{III.3}$$

where $\xi = \omega^2(\Delta t)^2$, $\eta = \Delta t\underset{\sim}{M}^{-1}\underset{\sim}{C}$, ω is any of the undamped structural resonant frequencies, and Δt is the time step size. The characteristics of the three roots of this equation (one real, two complex conjugates) can be investigated in terms of a single degree-of-freedom resonance and a damping factor in order to determine the extent of artificial damping and periodicity error. Figure 1 shows the values of the moduli of these roots, for the case of zero damping, plotted against the variable $\omega^2(\Delta t)^2$. The extraneous root [5], R_3 , causes the greater artificial damping (no artificial damping corresponds

to $R_i = 1$) but it seems clear that at least fifteen time steps (Levy and Kroll [13] suggest thirty) are required to maintain the modal amplitude at anywhere near the correct value. Figure 2 shows the same roots, plotted against smaller values of $\omega^2(\Delta t)^2$, showing that the extraneous root becomes the major factor in the artificial damping of the lower modes of structural response.

As an example of the choice of time step size to be used in conjunction with the Houbolt operator, consider the spherical shell cap under a point load at the apex as analyzed by McNamara and Marcal [4]. The load is applied suddenly and maintained at a constant value throughout the analysis. Although the shell is so thin and shallow that geometric nonlinearity dominates the dynamic response, some insight into the effect of artificial damping and periodicity error can be gained by comparing the time step used in the nonlinear analysis ($\Delta t = 10^{-5}$ seconds) with the periods of vibration for the lowest (linear) modes. Such a comparison is given in Table I. Noting that the abscissa of Fig. 2 can be written as $(\Delta t/2\pi T)^2$, where T is the period of

TABLE I

Spherical Shell Cap Modes and Periods

Mode	Frequency, Hz	Period, μsec	Δt/Period
1	892.2	1122.0	.00892
2	1165.0	858.0	.01165
3	1853.0	540.0	.01853
4	3092.0	323.5	.03092

vibration, the error due to artificial damping after 1000 time steps for the fourth mode response is about two percent. This can be considered negligible. It is unlikely that modal response above the fourth mode will be adequately

represented, however, unless the geometric nonlinearities increase the periods significantly.

The Wilson Averaging operator has also been analyzed [6] for artificial damping and comparisons with the Houbolt operator are shown in Figs. 3 and 4. While there is still strong damping for higher modes, the increased accuracy of the Wilson Averaging operator is evident. It should be noted, at this juncture, that the Newmark generalized acceleration operator, with $\gamma = 1/2$, (a) has a zero extraneous root and (b) has no artificial damping, regardless of the value of β [6].

The periodicity error comparisons are given in Figs. 5 and 6. The best performance is again exhibited by the Newmark operator with $\beta = 1/4$ (note that $\beta = 0$ causes the computed periods to be smaller than the exact periods, in contrast to the other integration operators). Of the two operators with nonvanishing extraneous roots, the Wilson Averaging operator is again superior. Figure 7 depicts the influence that real damping has on the artificial damping in the Houbolt operator. The effect on the complex conjugate roots is mild, since linearity of the change in damping is indicated for reasonable values of true damping. More disturbing is the effect displayed by the extraneous root, which shows a nonlinear <u>decrease</u> in damping as a function of <u>increased</u> true damping, even indicating an instability for large enough true damping.

IV. Stiffly Stable Methods

Gear [3] has suggested that the requirement for absolute stability for all components of the solution, regardless of the time constant of the component, is too restrictive and has indicated a preference for "stiffly stable" methods, i.e., methods which have regions of stability sufficient to include frequencies up to the cut-off frequency. From Fig. 8, the characteristics of a stiffly stable method are: (a) stable and accurate solutions in the cross-hatched region A; (b) stable, but not necessarily accurate, solutions in the cross-hatched region B; and (c) solutions of questionable stability and accuracy elsewhere. Each particular stiffly stable method has its own generic parameters ξ , η and ζ that define the regions of relative accuracy and stability. The A-stable methods correspond to $\eta = 0$.

The argument for using stiffly stable methods has been made, for structural dynamics problems, by Jensen [7]. He suggests that the order of the stiffly stable method be varied, depending upon the particular problem being solved, in order to maintain accuracy for the important components of the solution. It should be noted that stiffly stable methods of high order have the disadvantage of implicit backward difference operators such as the Houbolt operator--namely, that considerable storage space is occupied by the vectors of past displacement, velocity or acceleration.

Linear multistep methods for a system of first-order ordinary differential equations

$$\dot{\underset{\sim}{u}} = \underset{\sim}{f}(\underset{\sim}{u},t) \tag{IV.1}$$

with initial conditions

$$\underset{\sim}{u}(t_o) = \underset{\sim o}{u} \tag{IV.2}$$

are written in the form [21]

$$\sum_{i=1}^{m} \alpha_i \, u_{n+i} = \Delta t \sum_{i=1}^{m} \beta_i \, \dot{u}_{n+i} \quad , \; n = -1 \; , \; 0 \; , \; 1 \; ,\ldots \tag{IV.3}$$

where Δt is the step size, $u_n = u(t_o + n\Delta t)$, and $\alpha_m \neq 0$. If $\beta_m = 0$, (IV.3) is an explicit operator for u_{n+m} and is referred to as a predictor; if $\beta_m \neq 0$, (IV.3) is an implicit operator and is referred to as a corrector. Stiffly stable methods fall into the latter class; therefore, it is of interest to find out how they compare with implicit, unconditionally stable operators, such as the Newmark, Houbolt and Wilson Averaging operators.

In order to make this comparison, the equations of motion, (II.1), must be transformed from n second-order ordinary differential equations into 2n first-order ordinary differential equations. Jensen [7] has given such a decomposition that has the advantage of retaining the same size for the system of implicit equations, i.e., the additional n equations are already in upper or lower triangularized form.

Thus, define

$$v(t) \;=\; M \, \dot{u}(t) + (C + \gamma \Delta t K) u(t) \quad , \tag{IV.4}$$

where $\gamma = \beta_m / \alpha_m$. Then

$$\dot{v}(t) \;=\; F(t) - Ku(t) + \gamma \Delta t K \dot{u}(t) \quad . \tag{IV.5}$$

As an example, suppose that a single-step (m=2) method is being used. Then

$$\begin{Bmatrix} u \\ - - \\ v \end{Bmatrix}_{n+1} = -\frac{\alpha_1}{\alpha_2} \begin{Bmatrix} u \\ - - \\ v \end{Bmatrix}_n + \frac{\beta_1}{\alpha_2} \Delta t \begin{Bmatrix} \dot{u} \\ - - \\ \dot{v} \end{Bmatrix}_n + \frac{\beta_2}{\alpha_2} \Delta t \begin{Bmatrix} \dot{u} \\ - - \\ \dot{v} \end{Bmatrix}_{n+1} \tag{IV.6}$$

or

$$\left\{\begin{matrix} u \\ - \\ v \end{matrix}\right\}_{n+1} = \alpha\left\{\begin{matrix} u \\ - \\ v \end{matrix}\right\}_n + \beta\Delta t\left\{\begin{matrix} \dot{u} \\ - \\ \dot{v} \end{matrix}\right\}_n + \gamma\Delta t\left\{\begin{matrix} \dot{u} \\ - \\ \dot{v} \end{matrix}\right\}_{n+1} . \qquad \text{(IV.7)}$$

The governing system to be solved consists of equations (IV.4), (IV.5) and (IV.7). Some care must be taken when evaluating the stability properties of this operator, however, in that the starting procedure must be taken into account. Since $u(t_o)$ and $\dot{u}(t_o)$ are the prescribed initial conditions, the initial values for $v(t_o)$ and $\dot{v}(t_o)$ can be found from (IV.4) and (IV.5):

$$v(t_o) = M\dot{u}(t_o) + (C + \gamma\Delta tK)u(t_o) \qquad \text{(IV.8)}$$

and

$$\dot{v}(t_o) = F(t_o) - Ku(t_o) + \gamma\Delta tK\dot{u}(t_o) . \qquad \text{(IV.9)}$$

Therefore, the second of the partitioned equations (IV.7) must be rewritten, in general,

$$\begin{aligned} v_{n+1} &= \gamma\Delta t\dot{v}_{n+1} + \beta\Delta tF_n \\ &\quad + (\alpha M + \beta\gamma\Delta t^2K)\dot{u}_n \\ &\quad + (\alpha C + (\alpha\gamma-\beta)\Delta tK)u_n . \end{aligned} \qquad \text{(IV.10)}$$

The governing system to be solved can then be written in the Lax-Richtmyer form (see (II.2)) for direct integration operators where

$$[K_1] = \begin{bmatrix} \mathbf{I} & 0 & -\gamma\Delta t\mathbf{I} & 0 \\ 0 & \mathbf{I} & 0 & -\gamma\Delta t\mathbf{I} \\ -\mathbf{C}-\gamma\Delta t\mathbf{K} & \mathbf{I} & -\mathbf{M} & 0 \\ \mathbf{K} & 0 & -\gamma\Delta t\mathbf{K} & \mathbf{I} \end{bmatrix} , \qquad \text{(IV.11)}$$

$$[K_o] = \begin{bmatrix} \alpha\mathbf{I} & 0 & \beta\Delta t\mathbf{I} & 0 \\ \alpha\mathbf{C}+(\alpha\gamma-\beta)\Delta t\mathbf{K} & 0 & \alpha\mathbf{M}+\beta\gamma\Delta t^2\mathbf{K} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \qquad \text{(IV.12)}$$

and

$$\{F\}^T = \left\langle 0,\ \beta\Delta t\mathbf{F}_n,\ 0,\ \mathbf{F}_{n+1} \right\rangle , \qquad \text{(IV.13)}$$

where $\mathbf{I}$ is the identity matrix. Note that

$$\bar{\mathbf{u}}^T = \left\langle \mathbf{u},\ \mathbf{v},\ \dot{\mathbf{u}},\ \dot{\mathbf{v}} \right\rangle . \qquad \text{(IV.14)}$$

Without too much difficulty the inverse of (IV.11) can be found and premultiplied by (IV.12). Then the amplification matrix is given by

$$[A] = \begin{bmatrix} D^{-1}(\alpha M+\alpha\gamma\Delta tC-\beta\gamma\Delta t^2K) & 0 & D^{-1}(\alpha\gamma+\beta)\Delta tM & 0 \\ \alpha C-\beta\Delta tK & 0 & \alpha M & 0 \\ D^{-1}(\alpha\gamma+\beta)\Delta tK & 0 & D^{-1}(\alpha M-\beta\Delta tC-\beta\gamma\Delta t^2K) & 0 \\ -\alpha K & 0 & -\beta\Delta tK & 0 \end{bmatrix}, \qquad \text{(IV.15)}$$

where $D = M+\gamma\Delta tC+\gamma^2\Delta t^2K$. This expression confirms the fact that, if $\gamma = 0$ and the mass matrix is diagonal, the procedure is explicit. With $\gamma = 0$ and a distributed mass matrix, the operator remains implicit, since the mass matrix must be inverted.

The spectral character of this operator can be investigated by finding the eigenvalues of the amplification matrix as function of the structural frequencies.

The characteristic equation for A is

$$\lambda^2[D^{-1}(\alpha M+\alpha\gamma\Delta tC-\beta\gamma\Delta t^2K) - \lambda]\,[D^{-1}(\alpha M-\beta\Delta tC-\beta\gamma\Delta t^2K) - \lambda] + \lambda^2 D^{-1}KD^{-1}M(\alpha\gamma+\beta)^2\Delta t^2 = 0 \ . \qquad \text{(IV.16)}$$

Two of the eigenvalues are therefore seen to vanish, indicating the presence of extraneous roots [5] in the operator, as might have been expected. For the case of zero damping, the remaining two roots are complex conjugates, given by

$$\lambda_{1,2} = D^{-1}(\alpha M-\beta\gamma\Delta t^2K \pm i\Delta t(\alpha\gamma+\beta)\sqrt{K\,M}\ ) \ , \qquad \text{(IV.17)}$$

or, in terms of the structural frequencies,

$$\lambda_{1,2} = \frac{\alpha-\beta\gamma\omega^2\Delta t^2 \pm i(\alpha\gamma+\beta)\omega\Delta t}{1+\gamma^2\omega^2\Delta t^2} \quad . \tag{IV.18}$$

It would be interesting to compare this result, specialized for the trapezoidal rule $(\alpha = 1, \gamma = \beta = 1/2)$, with the conventional integration operators of structural dynamics. In this case,

$$\lambda_{1,2} = \frac{1-\frac{1}{4}\omega^2\Delta t^2 \pm i\omega\Delta t}{1+\frac{1}{4}\omega^2\Delta t^2} \quad . \tag{IV.19}$$

This result turns out to be identical [6] with the result for the Newmark generalized acceleration method with $\gamma = 1/2$, $\beta = 1/4$. Since the trapezoidal rule was shown by Dahlquist [9] to have the smallest asymptotic error of all order two methods of this class, in addition to being A-stable, it seems unlikely that much improvement can be made with other values of the parameters.

V. Nonlinear Problems

Other stiffly stable operators of higher order can be formulated [22-24], but since the storage requirements are large and since these higher order methods are also implicit, there seems to be little motivation for their study, unless it can be shown that the error in the Newmark method is excessive. Stricklin, et al. [25] have indicated a more serious problem--that the Newmark method degenerates when nonlinear problems are being analyzed, leading to unstable solutions. Both [25] and [4] have adopted the Houbolt method for this class of problems in order to take advantage of the artificial damping of spurious components in the solution due to the transpostion of nonlinear terms to the right-hand side of (II.1). Others [26,27,19] have suggested the use of either limited or unlimited modal superposition methods, even for nonlinear

problems. (It is worth noting that the Newmark operator has been used very successfully for stress formulations of the equations of motion [28].)

Boggs [29] has recently shown that the trapezoidal rule (or Newmark's method) is an effective procedure for solving nonlinear equations, provided that proper predictor-evaluation-corrector algorithms (PEC) are chosen. This means that a solution is predicted on the basis of stiffness matrices (initial stress, initial displacement, and small displacement matrices [30]) evaluated at the end of the last step; then, these matrices are re-evaluated on the basis of the predicted solution; finally, a corrected solution is sought. Boggs has found that an explicit predictor, an evaluation based on the predicted solution, and the trapezoidal rule corrector proves to be adequate. He also explored iterative methods which avoided the inversion of the Jacobian (iterative explicit).

In another recent publication. Weeks [31] has evaluated both the trapezoidal rule and the Houbolt operator (as well as central differencing) for geometrically nonlinear dynamic structural response problems. He found that a Newton Raphson iterative technique for both operators led to adequate results, and that the pseudo-load extrapolation procedure [25] caused instabilities for unconditionally stable operators when larger time steps are used. The extra storage and cost associated with the Newton Raphson technique makes its use in practical situations doubtful.

From all available evidence, then, it would seem that the trapezoidal, or Newmark, operator is the most attractive direct integration operator for both linear and nonlinear problems. The suggestions of Boggs [29] have very nearly been applied by McNamara and Marcal [4], who use an implicit predictor (in this case, the Houbolt method), evaluate on the basis of the predicted solution, then apply a "load correction" for the next implicit prediction,

based on the residual error from the equations of motion. This same procedure should also be applicable in conjunction with the trapezoidal rule and would seem to represent the optimum choice of a direct integration operator, considering economy, accuracy and stability.

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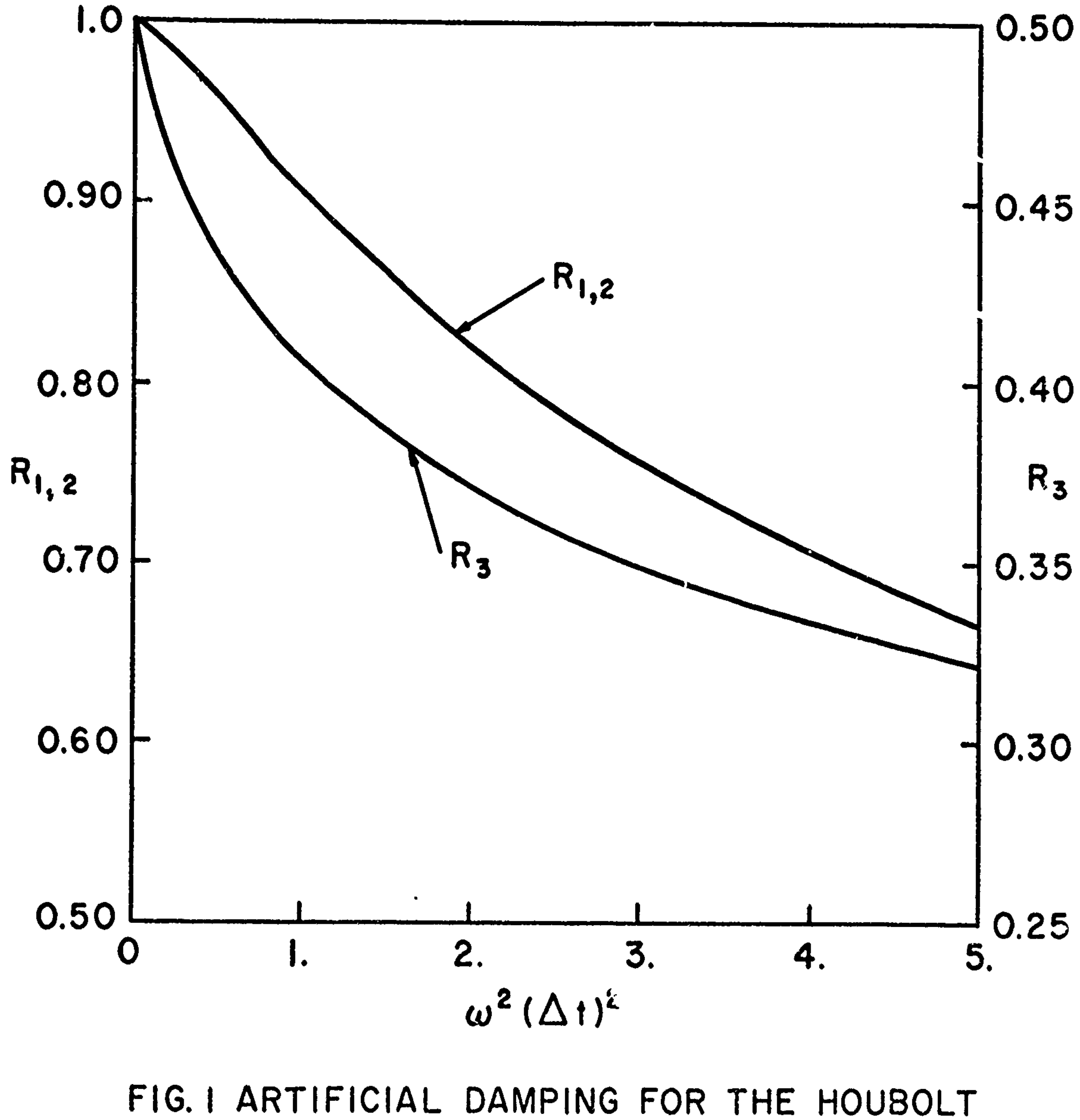

FIG. I ARTIFICIAL DAMPING FOR THE HOUBOLT OPERATOR SPECTRUM.

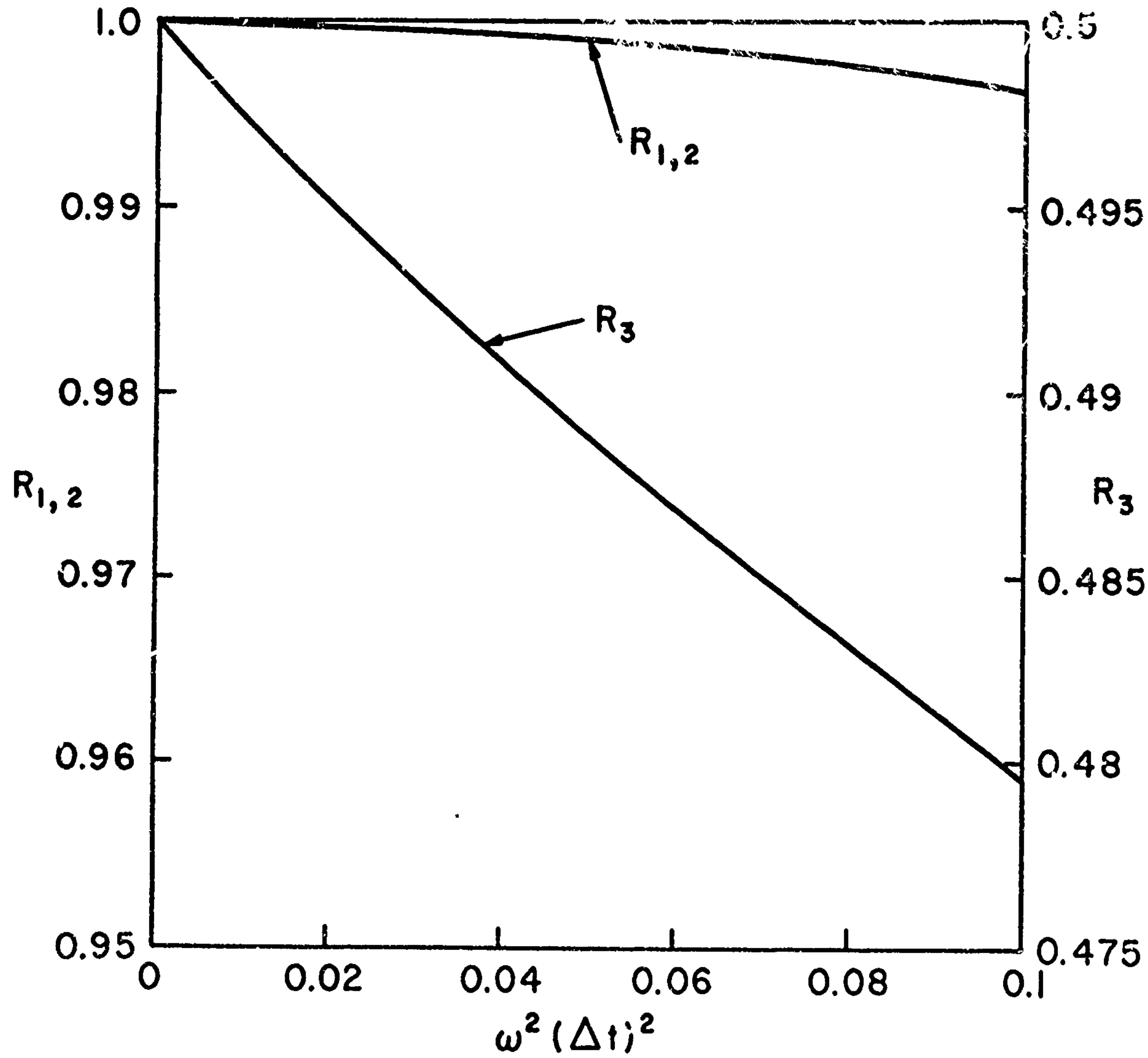

FIG. 2 EXTRANEOUS ROOT ARTIFICIAL DAMPING FOR THE HOUBOLT OPERATOR SPECTRUM.

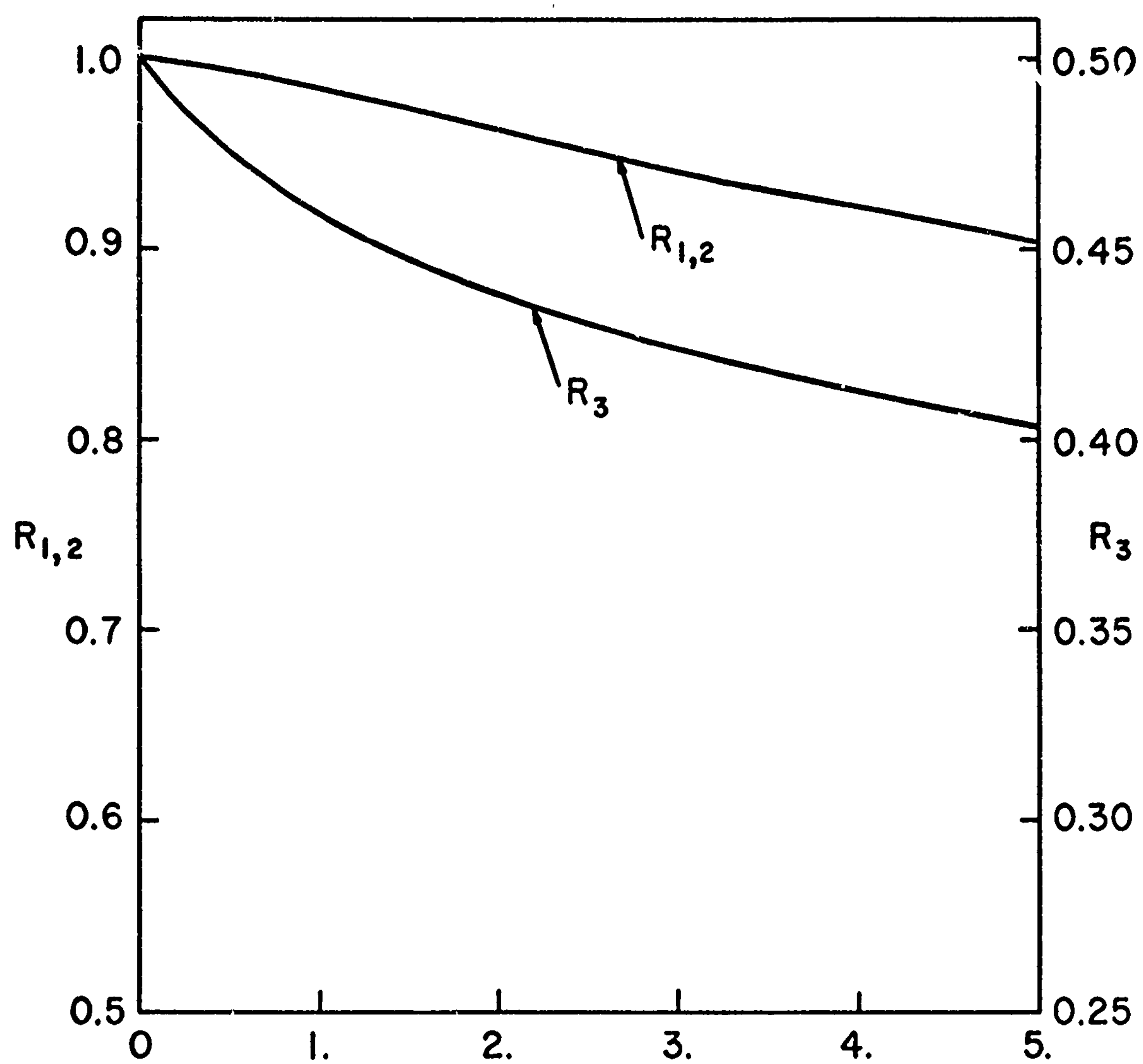

FIG. 3 ARTIFICIAL DAMPING FOR THE WILSON AVERAGING OPERATOR SPECTRUM.

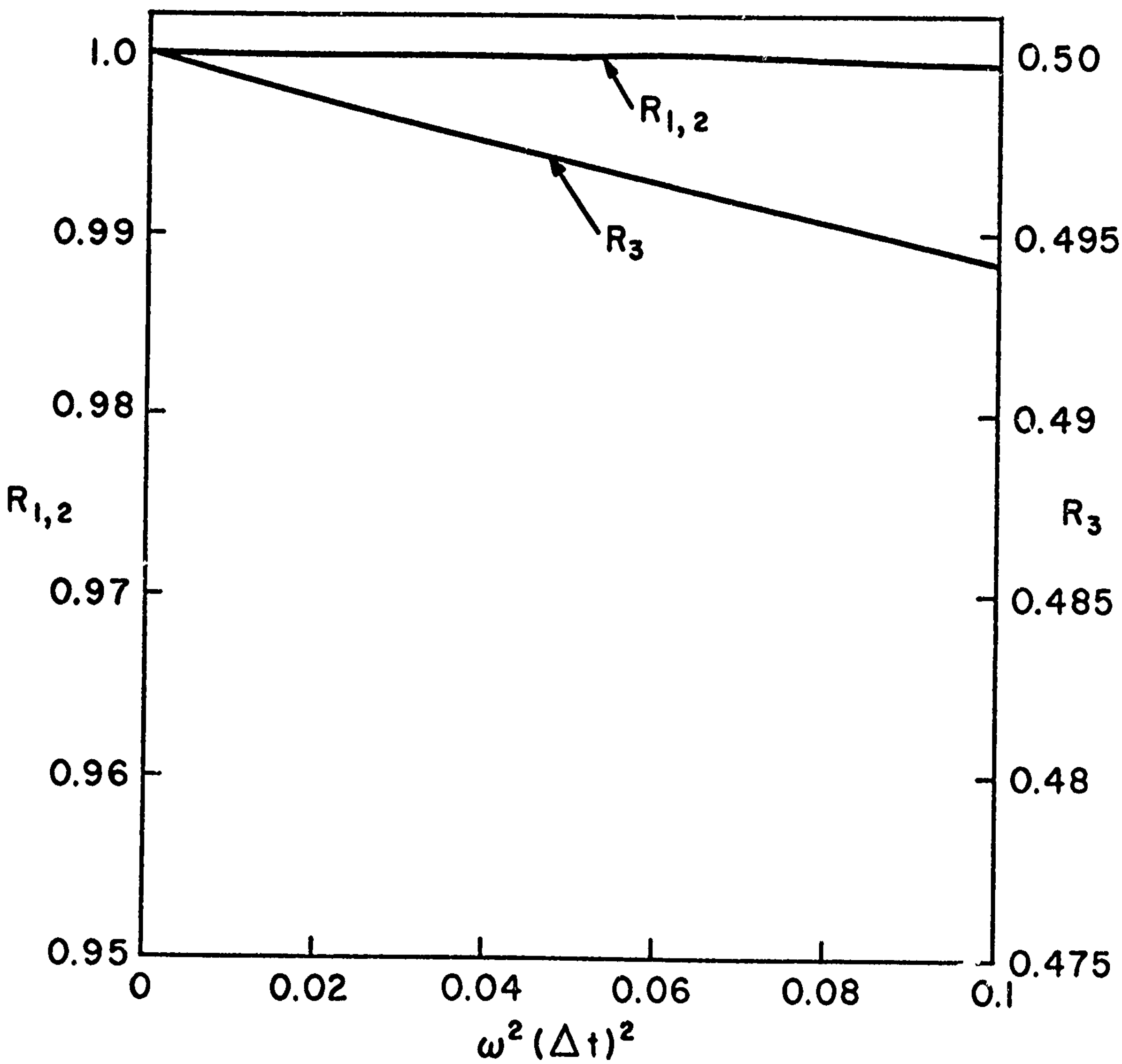

FIG. 4 EXTRANEOUS ROOT ARTIFICIAL DAMPING FOR THE WILSON AVERAGING OPERATOR SPECTRUM.

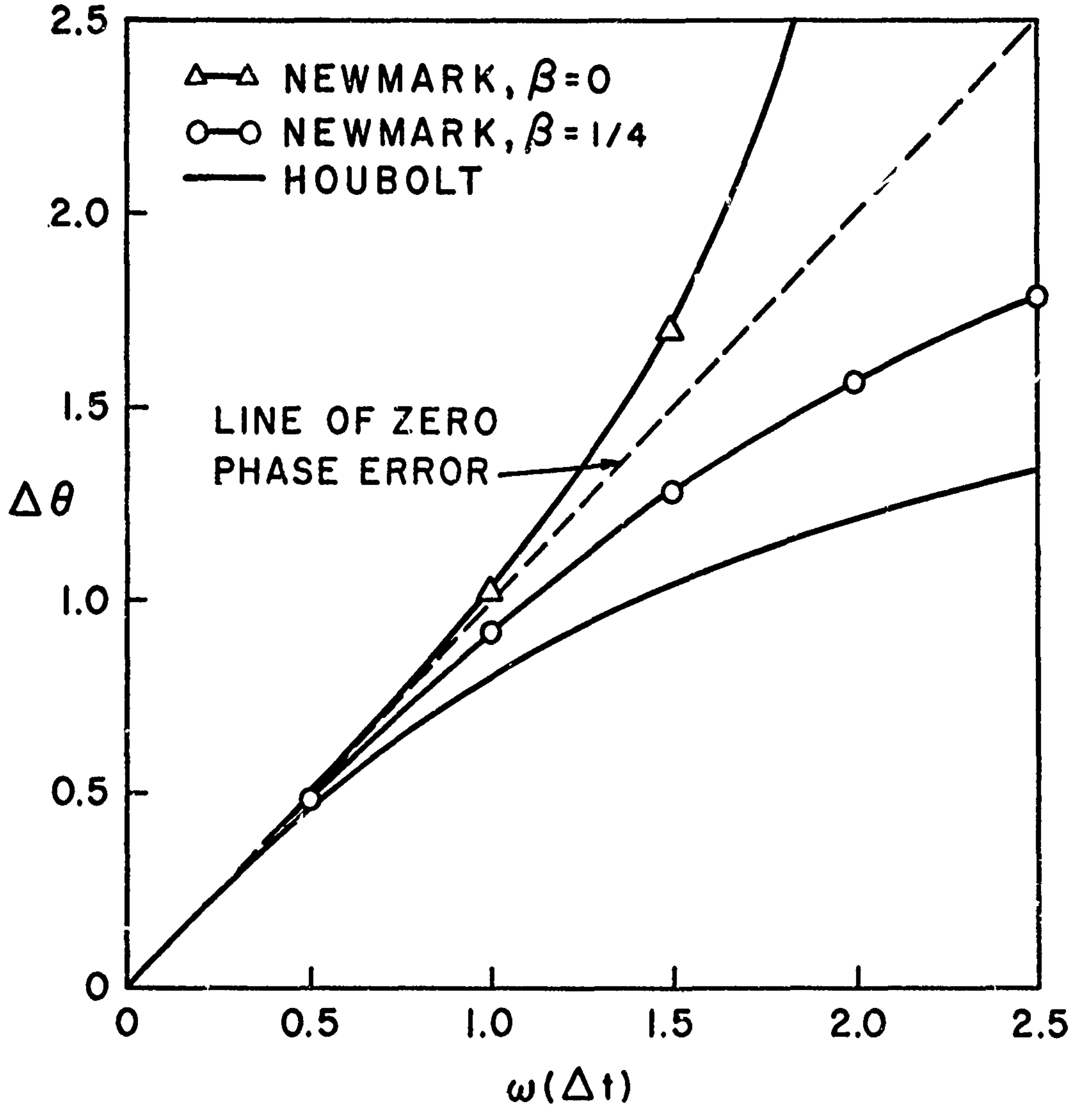

FIG. 5 PERIODICITY ERROR COMPARISON FOR NEWMARK AND HOUBOLT OPERATORS.

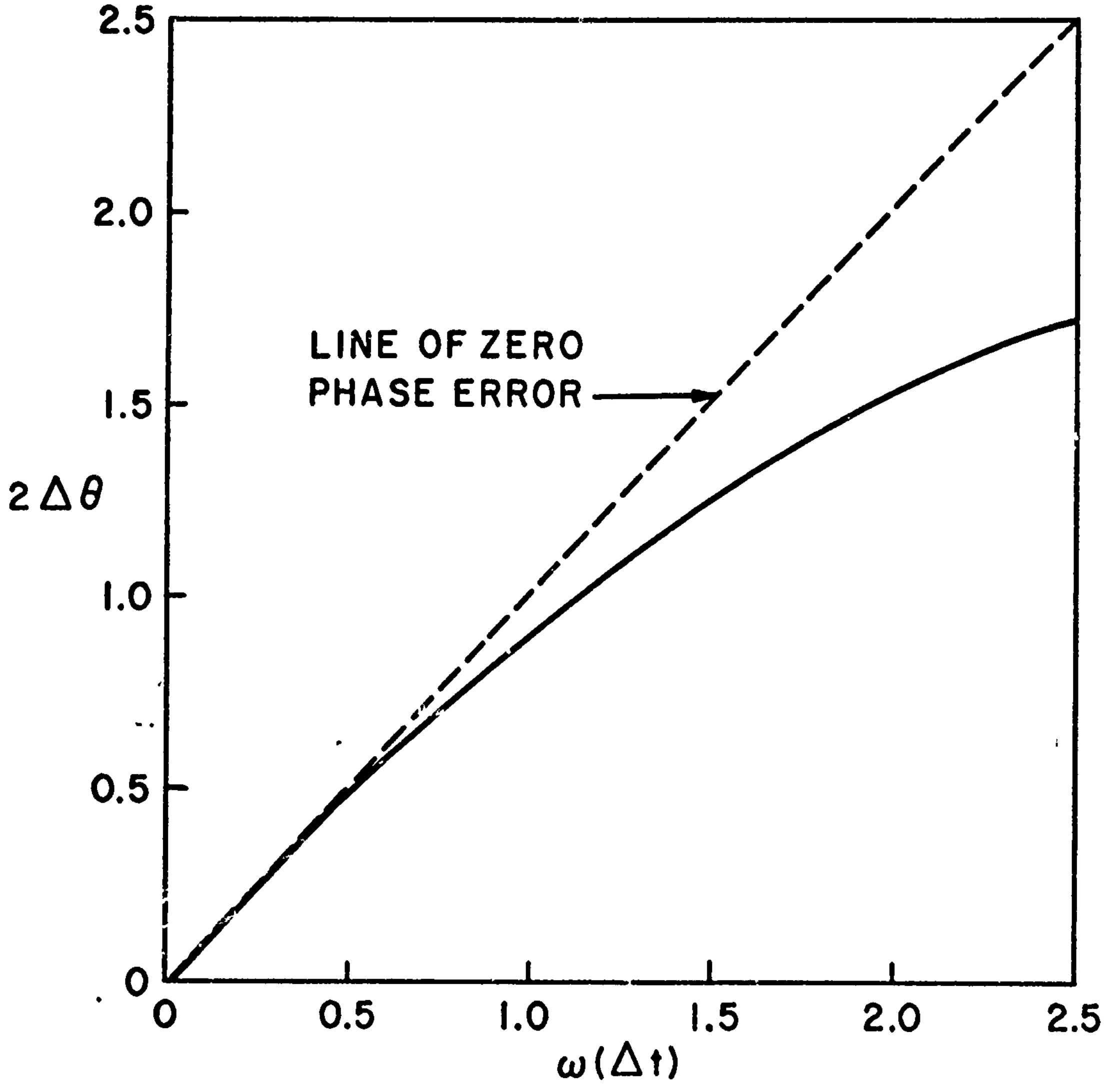

FIG. 6 PERIODICITY ERROR FOR THE WILSON AVERAGING OPERATOR.

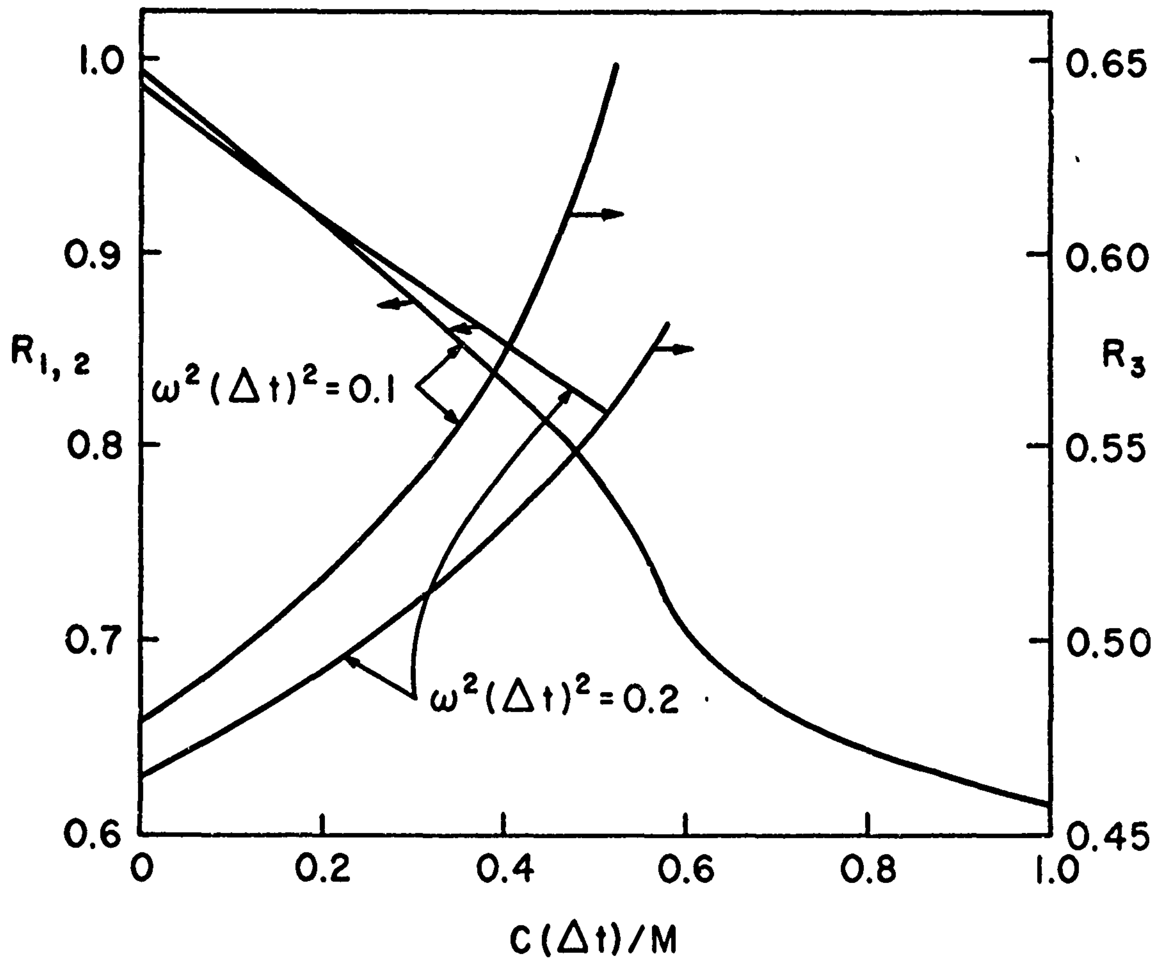

FIG. 7 EFFECT OF TRUE DAMPING ON ARTIFICIAL DAMPING, HOUBOLT OPERATOR.

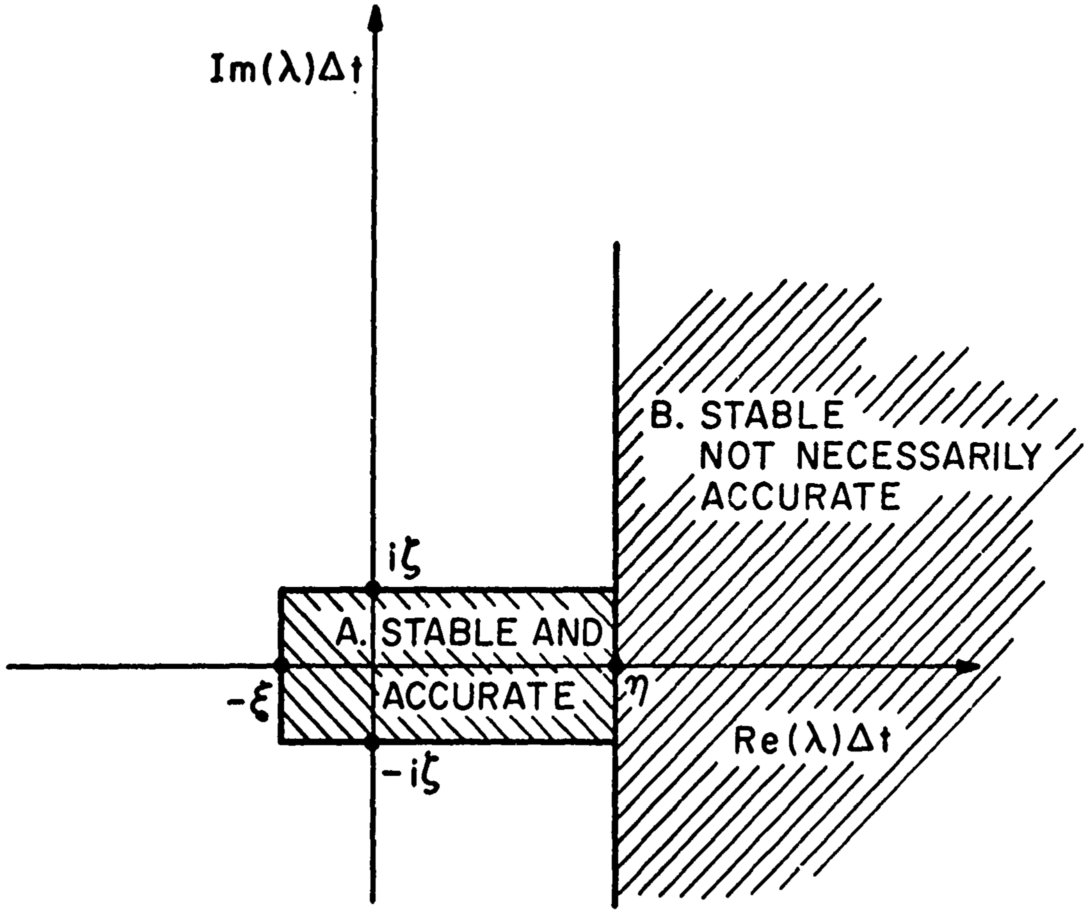

FIG. 8 STABILITY REGION FOR STIFFLY STABLE METHODS IN THE COMPLEX $\lambda \Delta t$ PLANE.